Chapter 2: Orthogonality & isotropy

Exercise 3. Let E be a $\mathbb{R}$-vector space of dimension $n < \infty$. Determine all $\varphi \in \mathscr{S}(E)$ (*i.e.* symmetric bilinear forms) such that $n = \dim\ker(\varphi) + 1$.

Solution of exercise 3. $\operatorname{rank}(\varphi) = 1$, for

$$\mathscr{J}_\varphi : E \longrightarrow E^\star$$
$$u \longmapsto j_u,$$

we have $\operatorname{Ker}(\varphi) = \operatorname{Ker}\mathscr{J}_\varphi$ has a basis consisting of $n-1$ vectors, say $u_2, \ldots, u_n$. Let $u_1 \notin \operatorname{Ker}(\mathscr{J}_\varphi)$, let $c = q(u_1)$. We have $c \neq 0$ since $\varphi \neq 0$. Define $\ell_1(u_i) = \delta_{i1}$, $\ell_2(u_i) = \delta_{i1}c$, then $\varphi(u,v) = \ell_1(u)\ell_2(v)$. □

Exercise 4. Let E be the vector space:

$$E := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{M}_2(\mathbb{R}) : a - d = 0 \right\}$$

Let also $J := \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ and define $q : E \longrightarrow \mathbb{R}$ by

$$q(A) := \operatorname{Tr}(AJA), \quad A \in E.$$

1. Prove q is a quadratic form and determine its polar form.
2. Prove that $\mathscr{B} := \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right)$ is a basis of E,
3. Give the matrix of q in the basis $\mathscr{B}$.
4. Determine the rank, the kernel and the signature of q. Is it positive? negative? definite?
5. Determine the orthogonal for q of $\operatorname{Span}(I_2)$.

Solution of exercise 4.

1. $q(\lambda A) = \operatorname{Tr}(\lambda^2 AJA) = \lambda^2 \operatorname{Tr}(AJA)$. $\varphi_q(A_1, A_2) = \frac{1}{2}(\operatorname{Tr}((A_1 + A_2)J(A_1 + A_2)) - \operatorname{Tr}(A_1JA_1) - \operatorname{Tr}(A_2JA_2)) = \frac{1}{2}(\operatorname{Tr}(A_1JA_2) + \operatorname{Tr}(A_2JA_1))$. φ_q is symmetric and bilinear.
2. Easy to check.
3. Set $v_1 := E_{11} + E_{22} = \mathrm{Id}$, $v_2 := E_{12}$ and $v_3 := E_{21}$. Then $\varphi_q(v_1, v_1) = 0$, $\varphi_q(v_1, v_2) = 1$, $\varphi_q(v_1, v_3) = 1$; $\varphi_q(v_2, v_2) = 0$, $\varphi_q(v_2, v_3) = 0$; $\varphi_q(v_3, v_3) = 0$. We have $\mathscr{M}_{\mathscr{B}}(\varphi_q)$ is the following:

$$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

4. $\text{rank}(q) = \text{rank}(\mathscr{M}_{\mathscr{B}}(\varphi_q)) = 2$. $\text{Ker}(q) = \{u \in E \mid \varphi_q(u,v) = 0\ \forall v \in E\}$, $\text{Ker}(q) = \text{Span}_{\mathbb{R}}(v_2 - v_3)$. Let $v = x_1v_1+x_2v_2+x_3v_3$, then $\varphi_q(v,v)_{\mathscr{B}} = (x_1\ x_2\ x_3)\begin{pmatrix}0&1&1\\1&0&0\\1&0&0\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} = 2x_1(x_2+x_3) = \frac{1}{2}[(x_1+x_2+x_3)^2-(x_1-x_2-x_3)^2]$.
It has signature $(1,1)$ thus not definite.

5. $\text{Span}(\text{Id})^{\perp} = \text{Span}(v_1)^{\perp} = \{v \in E \mid \varphi_q(v,v_1) = 0\}$. $\varphi_q(v,v_1)_{\mathscr{B}} = (x_1\ x_2\ x_3)\begin{pmatrix}0&1&1\\1&0&0\\1&0&0\end{pmatrix}\begin{pmatrix}1\\0\\0\end{pmatrix} = x_2 + x_3 = 0$, thus $\text{Span}(v_1)^{\perp} = \text{Span}(v_2 - v_3, v_1)$.

□

Remark. **Question**: Which step has problem in the following: By the standard process of finding the eigenspaces, we have

$$\begin{pmatrix}0&1&1\\1&0&0\\1&0&0\end{pmatrix} = PJ^tP \quad \text{where} \quad P = \begin{pmatrix}0&-\sqrt{2}/2&\sqrt{2}/2\\-1/\sqrt{2}&1/2&1/2\\1/\sqrt{2}&1/2&1/2\end{pmatrix} =: (u_1,u_2,u_3), \quad J = \begin{pmatrix}0&0&0\\0&-\sqrt{2}&0\\0&0&\sqrt{2}\end{pmatrix},$$

$q(a_1u_1 + a_2u_2 + a_3u_3) = (a_1,a_2,a_3)^tP\mathscr{M}_{\mathscr{B}}(\varphi_q)P\begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix} = -\sqrt{2}a_2^2 + \sqrt{2}a_3^2$, replace u_2 by $\sqrt[4]{2}u_2$, u_3 by $\sqrt[4]{2}u_3$, then $q(a_1u_1 + a_2u_2' + a_3u_3') = -a_2^2 + a_3^3$. q has signature $(1,1)$, and it is not definite.

Exercise 9. Consider the quadratic form q on $\mathbf{C}^3$ defined by

$$q(u_1,u_2,u_3,u_4) = u_1u_2 + u_2u_4 - u_3u_4 - 2u_1u_4 - 2u_2u_3 - u_1u_3.$$

1. Prove, without reducing q, that q has maximal rank,
2. Give an orthogonal basis for q.

Solution of exercise 9.

1. The associated matrix is $\begin{pmatrix}0&\frac{1}{2}&-\frac{1}{2}&-1\\\frac{1}{2}&0&-1&\frac{1}{2}\\-\frac{1}{2}&-1&0&-\frac{1}{2}\\-1&\frac{1}{2}&-\frac{1}{2}&0\end{pmatrix}$.

2.
$$\begin{aligned}q(u_1,u_2,u_3,u_4) &= u_1u_2 + u_2u_4 - u_3u_4 - 2u_1u_4 - 2u_2u_3 - u_1u_3\\&= u_1(u_2 - u_3 - 2u_4) + u_2u_4 - u_3u_4 - 2u_2u_3\\&= -\frac{1}{2}u_1^2 + \frac{1}{2}(u_1 + u_2 - u_3 - 2u_4)^2 - \frac{1}{2}(u_2 - u_3 - 2u_4)^2 + u_2u_4 - u_3u_4 - 2u_2u_3\end{aligned} \tag{1}$$

□

Exercise 10. Let E be a $\mathbb{R}$-vector space of dimension $n \geq 2$ and $f_1, f_2 \in E^\star$ be linearly independent. Let $q : E \to \mathbb{R}$ be defined by $q(u) = f_1(u)f_2(u)$ for all $u \in E$.

1. Show q is a quadratic form on E,
2. Determine the rank and the signature of q.

Solution of exercise 10.

1. $\varphi_q(u,v) = \frac{1}{2}(f_1(u+v)f_2(u+v) - f_1(u)f_2(u) - f_1(v)f_2(v)) = \frac{1}{2}(f_1(u)f_2(v) + f_1(v)f_2(u))$. The rest are easy to check.
2. $q = \frac{1}{4}((f_1+f_2)^2 - (f_1-f_2)^2)$. Set $\ell_1 := f_1 + f_2$, $\ell_2 = f_1 - f_2$, $(f_1\ f_2) = A(e_1^*\ e_2^*)$ then $(\ell_1\ \ell_2) = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} A(e_1^*\ e_2^*)$, since f_1, f_2 are linearly independent, A is invertible, thus $\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} A$ invertible and ℓ_1, ℓ_2 are linearly independent. Therefore q has signature $(1,1)$ and rank 2.

□

Exercise 11. Let $q \in \mathcal{Q}\left(\mathbb{R}^3\right)$ which matrix in the canonical basis of $\mathbb{R}^3$ is

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 5 \end{pmatrix}.$$

1. Determine the rank and the kernel of q,
2. What is the signature of q ?
3. Determine a basis of the orthogonal of $G := \mathrm{Span}((0,1,0),(1,0,1))$. Do we have $\mathbb{R}^3 = G \oplus G^\perp$?

Solution of exercise 11.

1. $\varphi_q(v,v)_{\text{canonical basis}} = (x_1\ x_2\ x_3)\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 5 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = x_1^2 + 2x_1x_2 + 2x_1x_3 + 6x_2x_3 + 2x_2^2 + 5x_3^2 = \ldots = (x_1 + x_2 + x_3)^2 + (x_2 + 2x_3)^2$. $\mathrm{rank}(q) = 2$. $\mathrm{Ker}(q) = \{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \mid \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 5 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0\} = \mathrm{Span}(\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix})$.
2. $(2,0)$.
3. $v = (x_1, x_2, x_3) \in G^\perp$ if: $(x_1\ x_2\ x_3)\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 5 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = x_1 + 2x_2 + 3x_3 = 0$, and $(x_1\ x_2\ x_3)\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 5 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = x_1 + 2x_2 + 3x_3 = 0$. Thus $G^\perp = \mathrm{Span}(\begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix})$. $\begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix} \in G \cap G^\perp$ so they are not in direct sum.

□

Exercise 13. Determine the signature of the quadratic form $q \in \mathcal{Q}(\mathbb{R}^n)$ whose matrix in the canonical basis of $\mathbb{R}^n$ is given by

$$(\min(i,j))_{1\leq i,j\leq n} = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & 2 & 2 & \cdots & 2 \\ 1 & 2 & 3 & \cdots & 3 \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & 2 & 3 & \cdots & n \end{pmatrix}.$$

Solution of exercise 13. Hint: Write it as sum of rank 1 matrices and use Exercise 3. □